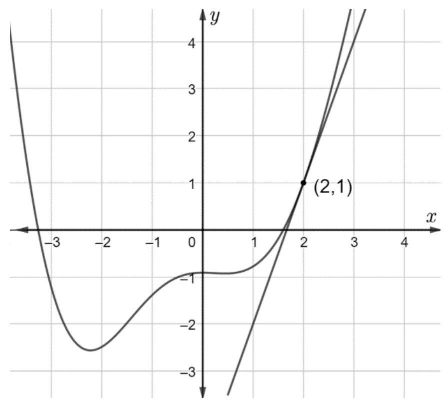
AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: 10.11	**Finding Taylor Polynomial Approximations of Functions – Day 2**	**Date:** April 9, 2020

Warm-Up

AP Practice Problem



A function f has derivatives of all orders for all values of x. A portion of the graph of f is shown above with the line tangent to the graph of f at $x = 2$. Let g be the function defined by $g(x) = 3 + \int_2^x f(t)\,dt$.

a.) Find the second degree Taylor polynomial for $g(x)$ centered at $x = 2$.

$$g(2) = 3 + \int_2^2 f(t)\,dt = 3 \qquad g'(2) = f(2) = 1 \qquad g''(2) = f'(2) = 3$$

$$P_2(x) = g(2) + g'(2)(x-2) + \frac{g''(2)(x-2)^2}{2!} = 3 + (x-2) + \frac{3(x-2)^2}{2!} = 3 + (x-2) + \frac{3}{2}(x-2)^2$$

b.) Does $g(x)$ have a local minimum, local maximum, or neither at $x = 2$? Give a reason for your answer.

$g'(2) = 1 \Rightarrow$ neither because $g'(2) \neq 0$, so $x = 2$ is not a critical point

c.) Consider the geometric series $\sum_{n=1}^{\infty} a_n$ where $a_1 = g'(2)$ and $a_2 = a_2 = P_2'(x) - 1$. Find $\sum_{n=1}^{\infty} a_n$ when $x = \frac{13}{6}$.

$$a_1 = g'(2) = 1 \qquad P_2(x) = 3 + (x-2) + \frac{3(x-2)^2}{2!} \qquad a_2 = P_2'(x) - 1 = 3(x-2)$$

$$r = \frac{a_2}{a_1} = \frac{3(x-2)}{1} = 3(x-2) \qquad x = \frac{13}{6} \Rightarrow r = 3\left(\frac{13}{6} - 2\right) = 3\left(\frac{1}{6}\right) = \frac{1}{2}$$

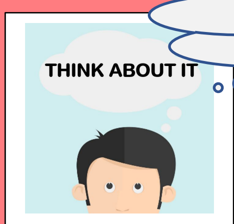
$$\sum_{n=1}^{\infty} a_n = \frac{1}{1 - \frac{1}{2}} = \frac{2}{2-1} = 2$$

Lesson Overview

WHAT WE ARE GOING TO DO	WHAT YOU SHOULD ALREADY KNOW
• Continue investigating the importance of Taylor polynomials and the role they play in the mathematics world. • Solve unique conceptual-based problems with Taylor polynomials.	• How to develop a Taylor polynomial from a given function. • The difference between a Tylor polynomial and a Maclaurin polynomial. • The components that make up the coefficients of the terms in a Taylor polynomial.

WHAT YOU WILL BE ABLE TO DO

Let f be a function having derivatives for all orders of real numbers. The 4th degree Taylor polynomial for f about $x = -2$ is given by $P_4(x) = -8 - \frac{5}{2}(x+2)^2 + k(x+2)^4$, where $k \neq 0$.



Question: If $f^{(4)}(-2) = \frac{2\pi}{3}$, find the value of k.

Guided Practice

Example 1: Find the third degree Taylor polynomial for $f(x) = \sin x$ centered at $x = \frac{\pi}{6}$.

Use this polynomial to approximate $\sin(0.2)$.

$$P_3(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)x + \frac{f''\left(\frac{\pi}{6}\right)}{2!}x^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}x^3$$

$$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x) \Rightarrow f'''(x) = -\cos(x)$$

$$P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2\cdot 2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2\cdot 3!}\left(x - \frac{\pi}{6}\right)^3$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$$

$\sin(0.2) \approx 0.19846\ldots$

Example 2: Find the second degree Maclaurin polynomial for $g(x)=e^{3x}$.

$$P_2(x)=g(0)+g'(0)x+\frac{g''(0)}{2!}x^2 \qquad g(x)=e^{3x}\Rightarrow g'(x)=3e^{3x}\Rightarrow g''(x)=9e^{3x}$$

$$P_2(x)=1+3x+\frac{9}{2}x^2$$

x	$f(x)$	$f'(x)$	$f''(x)$	$g(x)$	$g'(x)$
3	1	–2	7	4	–5

Example 3: The functions f and g are differentiable for all orders. The values of f, g, and selected derivatives of each are given in the table above at $x = 3$. For $n\geq 2$, the nth derivative of g at $x = 3$ is given by $g^{(n)}(3)=f^{(n-2)}(3)$. Find the third degree Taylor polynomial for $g(x)$ about $x = 3$.

$$P_3(x)=g(3)+g'(3)(x-3)+\frac{g''(3)}{2!}(x-3)^2+\frac{g'''(3)}{3!}(x-3)^3$$

$$P_3(x)=4+(-5)(x-3)+\frac{1}{2!}(x-3)^2+\frac{-2}{3!}(x-3)^3=4-5(x-3)+\frac{1}{2}(x-3)^2-\frac{1}{3}(x-3)^3$$

Example 4: A function $f(x)$ is not explicitly known but it is known that $f(2)=-7$ and $f'(2)=0$. Additionally, for $n>1$, $f^{(n)}(2)=\dfrac{n-1}{3^n}$. Find a 4$^{\text{th}}$ degree Taylor polynomial for $f(x)$ centered at $x=2$. Use this polynomial to approximate $f(3)$.

$$P_4(x)=f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3+\frac{f^{(4)}(2)}{4!}(x-2)^4$$

$$P_4(x)=-7+(0)(x-2)+\frac{\frac{2-1}{3^2}}{2!}(x-2)^2+\frac{\frac{3-1}{3^3}}{3!}(x-2)^3+\frac{\frac{4-1}{3^4}}{4!}(x-2)^4$$

$$P_4(x)=-7+\frac{1}{3^2 2!}(x-2)^2+\frac{2}{3^3 3!}(x-2)^3+\frac{3}{3^4 4!}(x-2)^4$$

$$P_4(3)=-7+\frac{1}{3^2 2!}(3-2)^2+\frac{2}{3^3 3!}(3-2)^3+\frac{3}{3^4 4!}(3-2)^4=-7+\frac{1}{3^2 2!}+\frac{2}{3^3 3!}+\frac{3}{3^4 4!}$$

$$f(3)\approx-7+\frac{1}{18}+\frac{1}{81}+\frac{1}{648}=-\frac{499}{72}=-6.9305\ldots$$

x	1	2
$f(x)$	–2	0
$f'(x)$	3	–1
$f''(x)$	5	–6

Example 5: The function h is defined by $h(x) = 4 + \int_2^{2x} f(t)\,dt$ where f is a twice differentiable function.

Selected values of f and its derivatives are given in the table above. Find the 2nd degree Taylor polynomial for $h(x)$ centered at $x = 1$.

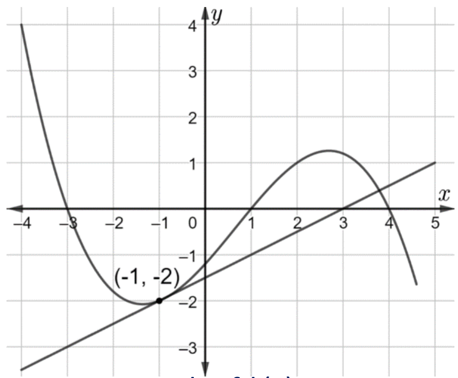
$h(1) = 4 \qquad h'(x) = 2f(2x) \qquad h''(x) = 2f'(2x)(2) = 4f'(2x)$

$$P_2(x) = h(1) + h'(1)(x-1) + \frac{h''(1)}{2!}(x-1)^2$$

$$P_2(x) = 4 + \left(2f(2)\right)(x-1) + \frac{4f'(2)}{2!}(x-1)^2$$

$$P_2(x) = 4 + \left(2(0)\right)(x-1) + \frac{4(-1)}{2!}(x-1)^2 = 4 - 2(x-1)^2$$

Check for Understanding



Graph of $k(x)$

Practice 1: A function k has derivatives of all orders for all values of x. A portion of the graph of k is shown above with the line tangent to the graph of k at $x=-1$. For $n \geq 2$, the nth derivative of $k(x)$ at $x=-1$ is given by:

$$k^{(n)}(-1)=\frac{n!}{n+1}.$$

Find the third degree Taylor polynomial for $k(x)$ about $c=-1$.

$$P_3(x)=k(-1)+k'(-1)(x+1)+\frac{k''(-1)}{2!}(x+1)^2+\frac{k'''(-1)}{3!}(x+1)^3$$

$$P_3(x)=(-2)+\frac{1}{2}(x+1)+\frac{\frac{2!}{2+1}}{2!}(x+1)^2+\frac{\frac{3!}{3+1}}{3!}(x+1)^3$$

$$P_3(x)=-2+\frac{1}{2}(x+1)+\frac{1}{3}(x+1)^2+\frac{1}{4}(x+1)^3$$

Practice 2: The functions f, f', and f'' are each continuous and differentiable. The nth derivative of f is given by $f^{(n)}(1)=\sum_{i=0}^{\infty}12\left(\frac{n+1}{5}\right)^i$ when $0\le n\le 3$. Find the third degree Taylor polynomial for $f(x)$ centered around $x=1$.

$$f(1)=f^{(0)}(1)=\sum_{i=0}^{\infty}12\left(\frac{0+1}{5}\right)^i=\frac{12}{1-\frac{1}{5}}=\frac{60}{5-1}=15 \qquad \text{geometric series } a=12, r=\frac{1}{5}$$

$$f'(1)=f^{(1)}(1)=\sum_{i=0}^{\infty}12\left(\frac{1+1}{5}\right)^i=\frac{12}{1-\frac{2}{5}}=\frac{60}{5-2}=20 \qquad \text{geometric series } a=12, r=\frac{2}{5}$$

$$f''(1)=f^{(2)}(1)=\sum_{i=0}^{\infty}12\left(\frac{2+1}{5}\right)^i=\frac{12}{1-\frac{3}{5}}=\frac{60}{5-3}=30 \qquad \text{geometric series } a=12, r=\frac{3}{5}$$

$$f'''(1)=f^{(3)}(1)=\sum_{i=0}^{\infty}12\left(\frac{3+1}{5}\right)^i=\frac{12}{1-\frac{4}{5}}=\frac{60}{5-4}=60 \qquad \text{geometric series } a=12, r=\frac{4}{5}$$

$$P_3(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3$$

$$P_3(x)=15+(20)(x-1)+\frac{30}{2!}(x-1)^2+\frac{60}{3!}(x-1)^3=15+20(x-1)+15(x-1)^2+10(x-1)^3$$

Practice 3: The function g is continuous and has derivatives for all orders at $x=-1$. It is known that $g(-1)=7$ and for positive values of n, the nth derivative of g at $x=-1$ is defined as the piecewise function given below:

$$g^{(n)}(-1)=\begin{cases} n^2+1, & n \text{ is odd} \\ 0, & n \text{ is even}\end{cases}$$

a.) Find the 5th degree Taylor polynomial, $P_5(x)$, for g centered at $x=-1$.

$$P_5(x)=g(-1)+g'(-1)(x+1)+\frac{g''(-1)}{2!}(x+1)^2+\frac{g'''(-1)}{3!}(x+1)^3$$
$$+\frac{g^{(4)}(-1)}{4!}(x+1)^4+\frac{g^{(5)}(-1)}{5!}(x+1)^5$$

$$P_5(x)=7+(1^2+1)(x+1)+\frac{0}{2!}(x+1)^2+\frac{(3^2+1)}{3!}(x+1)^3+\frac{0}{4!}(x+1)^4+\frac{(5^2+1)}{5!}(x+1)^5$$

$$P_5(x)=7+2(x+1)+\frac{5}{3}(x+1)^3+\frac{13}{60}(x+1)^5$$

b.) Determine if $P_5(x)$ is increasing or decreasing at $x=-1$. Explain your reasoning.

$$P_5'(x)=2+5(x+1)^2+\frac{13}{12}(x+1)^4 \Rightarrow P_5'(-1)=2$$

$P_5(x)$ is increasing at $x=-1$ because $P_5'(-1)=2>0$

Practice 4: The fourth degree Taylor polynomial for $f(x)$ centered about $x=2$ is given by
$T_4(x)=2-3(x-2)+\frac{3(x-2)^2}{4}-\frac{4(x-2)^3}{9}+\frac{7(x-2)^4}{26}$. Find the value of $f'''(2)$.

$$-\frac{4}{9}=\frac{f'''(2)}{3!}\Rightarrow f'''(2)=-\frac{4\cdot 3!}{9}=-\frac{8}{3}$$

Practice 5: Consider the function $g(x)=e^{\frac{-2x}{3}}$. Find the coefficient of the x^{26} term in the Maclaurin polynomial of degree 26.

$$g'(0)=\left(-\frac{2}{3}\right)e^{\frac{-2(0)}{3}}=\left(-\frac{2}{3}\right) \qquad g''(0)=\left(-\frac{2}{3}\right)^2 e^{\frac{-2(0)}{3}}=\left(-\frac{2}{3}\right)^2$$

$$g^{(n)}(0)=\left(-\frac{2}{3}\right)^n \Rightarrow \frac{g^{(26)}(0)}{26!}=\frac{\left(-\frac{2}{3}\right)^{26}}{26!} \text{ is the coefficient of the } x^{26} \text{ term}$$

Debrief and Summary

ENDURING UNDERSTANDINGS	KEY TAKEAWAYS
Power series allow us to represent associated function on an appropriate interval.	• A Taylor polynomial allows us to approximate the values of more complicated function. • In many cases, as the degree of the Taylor polynomial increases, the polynomial will approach the original function over some growing interval. • Taylor polynomials can be extended to Taylor series and leads to the idea of power series.

COMMON ERRORS, MISCONCEPTIONS & PITFALLS

- Students sometimes forget that the coefficients of the terms of a Taylor polynomial are of the form $\frac{f^{(n)}(c)}{n!}$.
- Taylor polynomials can produce approximations, but we have not learned how "good" that are at approximating our desired function….yet!

AP Exam Practice

AP Practice Problem

A function f has derivatives of all orders at all real x values.

a.) Let $P_2(x)$ represent the 2[nd] degree Maclaurin polynomial for f. It is known that $f(0)=1$ and $f'(0)=0$. If $P_2(1)=\frac{1}{2}$, find $f''(0)$.

$$P_2(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2=1+\frac{f''(0)}{2!}x^2$$

$$P_2(1)=1+\frac{f''(0)}{2!}(1)^2=1+\frac{f''(0)}{2!}=\frac{1}{2}\Rightarrow\frac{f''(0)}{2!}=-\frac{1}{2}\Rightarrow f''(0)=-1$$

b.) Find $P_2'(0)$ and $P_2''(0)$. Does $P_2(x)$ have a relative minimum, relative maximum, or neither at $x = 0$? Give a reason for your answer.

$P_2(x)=1-\frac{1}{2}x^2$ $\qquad$ $P_2'(x)=-x\Rightarrow P_2'(0)=0$ $\qquad$ $P_2''(x)=-1\Rightarrow P_2''(0)=-1$

$P_2(x)$ has a relative maximum at $x=0$ because $P_2'(0)=0$ and $P_2''(0)<0$, concave down.

c.) Use $P_2(x)$ to approximate $f\left(\frac{1}{2}\right)$.

$$f\left(\frac{1}{2}\right)\approx P_2\left(\frac{1}{2}\right)=1-\frac{1}{2}\left(\frac{1}{2}\right)^2=1-\frac{1}{8}=\frac{7}{8}$$